Suggested reading: Meester, Chapter 1, §§1-3

Problems:

1. Suppose \( P(A) = 2/5 \), \( P(B) = 3/5 \) and \( P(A \text{ or } B) = 4/5 \). Find the probabilities of the following events.
   (a) Neither \( A \) nor \( B \) occurs.
   (b) Both \( A \) and \( B \) occur.
   (c) Exactly one of the two events occurs.

2. Let \( A \) and \( B \) be events with \( A \subseteq B \).
   (a) Show that \( P(A) \leq P(B) \).
   (b) In fact, show that \( P(B \setminus A) = P(B) - P(A) \).

3. Alice, Bob and Charlie take turns rolling a standard six-sided die, in the order ABCABC.… The first person to roll a six wins the game. Find each person’s chance of winning.

4. (a) Use set algebra to establish the inclusion-exclusion formula for three events:

\[
P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)
\]

*Hint.* Start with \( A \cup B \cup C = A \cup (B \cup C) \) and apply the two-event version. You will need to use the Distributive Law.

(b) You toss a fair coin six times. What is the probability that at some point you will have seen (at least) four consecutive heads?

5. Recall that a Vermont license plate consists of three letters (A,B,…,Z) followed by three digits (0,1,…,9). Suppose the characters are selected at random.
   (a) What is the chance that no pair of adjacent symbols are the same?
   (b) What is the chance that no three adjacent symbols are the same?

6. Recall that there are \( \binom{52}{5} = 2,598,960 \) possible poker hands. If you are dealt a hand from a well-shuffled deck, what is the probability that you have
   (a) No aces? (Check that your answer equals 35,673/54,145 as asserted in HW 1.7.)
   (b) At least one ace?
   (c) The ace of spades?
   (d) Exactly one ace?
7. Recall that in a bridge game, each of the players North, East, West and South is dealt 13 cards.

(a) What is the probability that North holds all four aces?
(b) What is the probability that one of the players holds all four aces?

8. People in Galileo’s time thought that when three dice were rolled, a sum of 9 and a sum of 10 ought to be equally likely, since each sum could be obtained in six ways, as shown below.

9 : 1 + 2 + 6, 1 + 3 + 5, 1 + 4 + 4, 2 + 2 + 5, 2 + 3 + 4, 3 + 3 + 3
10 : 1 + 3 + 6, 1 + 4 + 5, 2 + 4 + 4, 2 + 2 + 6, 2 + 3 + 5, 3 + 3 + 4

However, experience suggested that 10 was more likely to appear as a sum than 9. Sometime around 1642 (which predates the Pascal-Fermat correspondence), Galileo was able to explain this.

(a) By being a little more careful counting the ways the sums can occur, compute the two probabilities in question.
(b) Use R to simulate the probabilities. *Hint.* See the “R Tutorial” handout. In addition to the commands there, you might try the function `table(x)`.

If `nreps` is the number of repetitions in your simulation, what happens when you do `table(x)/nreps`?

9. Prove the binomial identity

$$\binom{n}{k} = \binom{n-k+1}{k} \binom{n}{k-1}$$

in the following two ways.

(a) By algebra with factorials.
(b) By counting the same thing two ways. (*Hint: try a committee story.*)