Problems:

1. In class we derived a formula for the row sums in Pascal’s triangle. We now explore a formula for the (partial) sums of the columns.
   (a) Verify that \( \binom{6}{3} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} \). Observe that you are finding an entry in the \( k = 3 \) column by summing the values above it in the \( (k - 1) = 2 \) column.
   (b) By “counting the same thing two ways,” prove that the general result is

   \[
   \binom{n}{k} = \sum_{m=k}^{n} \binom{m-1}{k-1}
   \]

   Above you illustrated the formula with \( n = 6, k = 3 \).
   
   Hint: How many ways are there to choose a \( k \)-element subset from \( \{1, 2, \ldots, n\} \) whose largest element is \( m \)? For what values of \( m \) is it possible to do so?

2. In the “Take 5” game from the New York Lottery, five winning numbers are selected randomly from a set of balls numbered 1, 2, \ldots, 39. Players mark their tickets by marking choosing numbers. Some people select their “lucky numbers” by playing family birthdays. Of course, this means there will be no number larger than 31.
   (a) What is the chance that the largest number drawn in the Lottery is 31?
   (b) What is the probability that no number larger than 31 is drawn (so the largest might be 25, for example)?

3. Recall that we defined independence of events \( A \) and \( B \) by the equation

   \[
P(AB) = P(A)P(B)
   \]

   Surely, if \( A \) and \( B \) are independent, then \( A \) and \( B^c \) are independent. But “surely” requires a proof. (Or, if it turns out that \( A \) and \( B^c \) are not independent, perhaps we might want to reconsider the definition!)
   (a) Assuming that \( A \) and \( B \) are independent, prove that \( A \) and \( B^c \) are independent.
   (b) You can now immediately conclude that \( A^c \) and \( B^c \) are also independent. Why?

4. The Vermont “Pick 3 Lottery” runs twice a day, 7 days a week. Each drawing is equally likely to produce any three-digit number from 000 through 999. What is the probability that none of the digits from the daytime drawing will appear in that evening’s drawing?
5. Consider a craps game played with six-sided dice, but instead of the labels 1, 2, 3, 4, 5, 6 appearing on each die, the labels 1, 1, 4, 4, 6, 6 appear on one die and 1, 2, 3, 5, 5, 6 appear on the other.

(a) Show that the same sums can appear with the new dice as with standard dice, and find the corresponding probabilities.

(b) Show that if you play craps with the new dice, the probability of winning the pass-line bet is 1/2.

6. Recall this game from HW#2: A, B and C roll a die one at a time in the order ABCABC . . . . The first person to roll a 6 wins the game. Suppose now that after a winner is determined, the two remaining players continue tossing in order, and the next person to roll a 6 is declared the runner-up.

(a) Find the probability that B wins and A is the runner up.

(b) Find the probability that A is the overall loser.

7. A chessboard has 64 squares (8 rows and 8 columns). We say two rooks on the board are “attacking” if they are in the same row or column. If eight rooks are placed on distinct squares at random, what is the probability that there is no pair of attacking rooks?

8. We saw that checking independence of two events involves one equation, while for three events it takes four equations. Show that for n events there are $2^n - n - 1$ equations to check.

9. Recall that the complementary probability for the birthday problem is given by

$$q_{23} = \frac{365!/(365 - 23)!}{365^{23}}$$

In class, we calculated this using a recursive formula. If you try to compute the above directly in R using the `factorial()` function, you will get overflow problems (try it so that you know what this looks like!). For use in such situations, R includes the `lfactorial()` function, which computes the natural logarithm of a factorial. Thus `lfactorial(365)` gives log(365!).

Use log properties for find an expression for log($q_{23}$) that you can evaluate using `lfactorial()` R, and then exponentiate the result to get $q_{23}$. (Recall that in R, `exp()` is the natural exponential function, and `log()` is the natural logarithm.)