Suggested reading: Review Meester, §§1.4, 1.5.

Problems:

1. Consider the experiment of rolling two dice. Let $A$ be the event “the first die shows 1, 2, or 3.” Let $B$ be the event “the first die shows 3, 4, or 5.” Let $C$ be the event “the sum of the rolls is 9.”

Show that $P(ABC) = P(A)P(B)P(C)$, but that no pair of the events is independent.

(In class, we saw that pairwise independence did not imply that 3-way multiplication would work; here we see that the other direction also fails).

2. Suppose that Urn #1 contains 3 red balls and 2 blue balls, while Urn #2 has 4 red and 7 blue balls. We select an urn by fair coin toss, and then choose a ball at random from that urn.

   (a) What is the probability that this procedure results in a blue ball from Urn #2?
   (b) What is the probability that a blue ball is removed?
   (c) Given that a blue ball is selected, what is the conditional probability that it comes from Urn #2?
   (d) Without replacing the first ball, a second ball is removed from the same urn, and it also turns out to be blue. What now is the (conditional) probability that the balls were chosen from Urn #2?

3. (a) True or False: $P(A | B) + P(A | B^c) = 1$.

   Show the statement is true for any events $A, B$ or exhibit a counterexample.

   (b) Give a formula for $P(A | B^c)$ in terms of $P(A), P(B)$ and $P(AB)$ only.

4. Let $D$ be an event with $P(D) > 0$. Events $A$ and $B$ are said to be conditionally independent given $D$ if

   $P(A \cap B | D) = P(A | D)P(B | D)$

   (Observe that this corresponds to independence under the probability measure $P_D$).

   Consider two tosses of a fair coin, with events $H_1 =$ “1st toss is heads”, $H_2 =$ “2nd toss is heads” and $D =$ “tosses are different”.

   (a) Show that $H_1$ and $H_2$ are (unconditionally) independent, but they are not conditionally independent given $D$.

   (b) Show that if $P(B \cap C) > 0$, then conditional independence of $A$ and $B$ given $C$ is equivalent to

   $P(A | B \cap C) = P(A | C)$. 