**MATH 310 Homework #7**  
(Due Wednesday, 31 October 2018)

**Suggested reading:** Meester, Chapter 2, §3 (up to Variance).

**Problems:**

1. *St. Petersburg Paradox.* Recall the coin-tossing game in which you win $2^k$ if your first head appears on the $k$th toss. Suppose that the house has a total of $2^m$, so that this is the most you could win, even if it takes more than $m$ tosses to produce a head. Show that the expected value of the game is now $(m + 1)$.

2. Consider a random variable $X$ whose probability mass function is described as follows:

   $$P(X = 0) = \frac{3}{4}; \quad P(X = k) = \frac{1}{2} \left(\frac{1}{3}\right)^k, \quad k = 1, 2, \ldots$$

   (a) Verify that this is a valid PMF.
   (b) Find $E[X]$.

3. The number of fish in a certain lake is given by a random variable $X$ having a Poisson distribution with parameter $\lambda$. Worried that there might be no fish at all, a statistician adds one fish to the lake. Let $Y = X + 1$ denote resulting number of fish.

   (a) Find $E[Y]$.
   (b) Find $E[1/Y]$.

4. You roll a fair tetrahedral (4-sided) die until the first time you observe a face that has already come up on a previous roll. Find the expected number of rolls required in two ways:

   (a) By using the tail sum formula.
   (b) By finding the probability mass function.

5. Let the random variable $N$ be the minimum of the two numbers obtained rolling a pair of tetrahedral dice.

   (a) Observe that the minimum of two numbers exceeds $x$ if and only if both numbers exceed $x$. Apply this idea to find $P(N > k)$ for $k = 0, 1, 2, \ldots$
   (b) Find $E[N]$ using the tail sum formula.
   (c) Let the random variable $M$ be the maximum of the two rolls, and let $S$ be the sum. Explain why $M + N = S$, and use this result to find $E[M]$. Check your answer using the PMF from HW 5.1(c).
   (d) Find a formula for the expected minimum when $d$ dice are rolled. What happens as $d \to \infty$? Explain why this makes intuitive sense.
6. Recall that the Vermont “Pick 4 Lottery” runs twice a day, 7 days a week. Each drawing is equally likely to produce any four-digit number from 0000 through 9999. On April 12, 2018, the daytime drawing was 6773 and the evening drawing was 7439. Observe that two digits, 7 and 3, appear in both. What is the expected number of different digits that will appear in both in the daytime and evening drawings?

7. From a well-shuffled deck, cards are dealt face up one at a time until all the aces have appeared.
   (a) What is the probability that all 52 cards are dealt?
   (b) What is the expected number of cards that are dealt?
   (c) Find the expected number of face cards (Jack, Queen or King) remaining in the deck.

   *Hint.* Think about techniques from ‘Large Urn’ problem!

8. The *high card points* in a bridge hand are computed by counting 4 points for each ace, 3 points for each king, 2 points for each queen and 1 point for each jack. Find the expected total number of high card points in a 13-card hand in the following two ways.
   (a) Find a representation in terms of hypergeometric random variables.
   (b) Appeal to symmetry.

9. People are interviewed one at a time until the first time someone has a birthday match with any of the earlier people.
   (a) Use R to apply the tail sum formula to compute the expected number of people that will be interviewed. If you look at our script on the web for the birthday problem, you can see how to construct a vector \( q \) in which \( q[r] \) gives the probability that there is no birthday match among \( r \) people.
   (b) Write a simulation to estimate the expectation. If you generate 366 random birthdays, you are guaranteed at least one match! The following commands may be useful:
      - \texttt{duplicated(v)} gives the elements of the vector \( v \) that are duplicates of earlier elements (i.e., elements with smaller subscripts);
      - \texttt{which(duplicated(v))} gives the positions where the duplicates occur;
      - \texttt{which(duplicated(v))}[1] gives the first of these positions.